

MINE 432: Industrial Automation and Robotics in Mining

Lecture 6

Modeling in Digital Form

PID controllers were introduced in Lecture 3 and their mathematical equations were presented and discussed. The problem with Eq. 3.1, 3.2, and 3.3 is that the value of p_o cannot be determined for each point in time and so, a new form of these equations is required. The form presented here allows development of control programs to operate in a computer environment with digital sampling of signals. The previous form established an analog mode of operation required for pneumatic and mechanical operation of control equipment.

Proportional Control

Recall that the analog relationship from Eq. 3.2 is:

$$p(t) = K_c \varepsilon(t) + p_o \quad (6.1)$$

where

$p(t)$ = Controller Output

$\varepsilon(t)$ = Error signal

p_o = Initial value of the Controller Output

The initial value with a digital signal cannot be established, so instead one must consider the difference between the current and previous controller output value:

$$\text{change in output} = K_c * \text{change in error}$$

so we get:

$$p(t) - p(t-1) = K_c * [\varepsilon(t) - \varepsilon(t-1)] \quad (6.2)$$

or by rearranging

$$p(t) = p(t-1) + K_c * [\varepsilon(t) - \varepsilon(t-1)] \quad (6.2a)$$

where

$p(t)$ = Current Controller Output

$p(t-1)$ = Previous Controller Output

$\varepsilon(t)$ = Current Error Signal

$\varepsilon(t-1)$ = Previous Error Signal

This subtle change in form allows the equation to be used for real-time digital application in a computer subroutine tied to a process, a final control element, and a measurement block as shown previously in Fig. 3.1. Take care to account for the difference in error, since by including only the current error value, one will actually implement Integral control, not Proportional control. Table 6.1 shows example data to demonstrate that these two equations (6.1 and 6.2a) are equivalent. The data are plotted in Fig. 6.1 to show this comparison in graphical form.

Table 6.1. Example data to demonstrate the equivalency of the difference form (Eq. 6.2a) to that of Eq. 6.1 for Proportional Control.
(where $K_c = 25$; $p_o = 325$; $A = 1$)

Step change: $\epsilon(t) = A$ where $A = 1.0$

t	$t-1$	$\epsilon(t)$	$\epsilon(t-1)$	$p(t)$ from Eq.6.2a	$p(t)$ from Eq. 6.1
0	-1	1	0	350	350
1	0	1	1	350	350
2	1	1	1	350	350
3	2	1	1	350	350
4	3	1	1	350	350
5	4	1	1	350	350

Ramp change: $\epsilon(t) = At$ where $A = 1.0$

t	$t-1$	$\epsilon(t)$	$\epsilon(t-1)$	$p(t)$ from Eq.6.2a	$p(t)$ from Eq. 6.1
0	-1	0	0	325	325
1	0	1	0	350	350
2	1	2	1	375	375
3	2	3	2	400	400
4	3	4	3	425	425
5	4	5	4	450	450

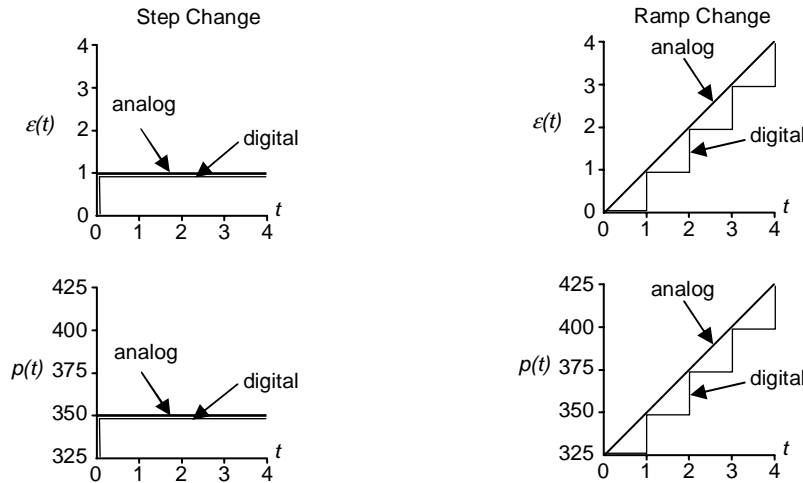


Fig. 6.1. Proportional controller response to an analog or digital error signal change.

Integral Control

Recall that the analog relationship from Eq. 3.2 for Integral Control is:

$$p(t) = \frac{1}{T_I} \int \epsilon(t) dt + p_o \quad (6.3)$$

where

- $p(t)$ = Controller Output
- $\varepsilon(t)$ = Error signal
- p_o = Initial value of the Controller Output

The initial value with a digital signal cannot be established, so instead one must consider calculating the area numerically:

so we get:

$$p(t) - p(t-1) = \frac{1}{T_i} * [\varepsilon(t) + \varepsilon(t-1)] * \frac{\Delta t}{2} \quad (6.4)$$

or by rearranging

$$p(t) = p(t-1) + \frac{1}{T_i} * [\varepsilon(t) + \varepsilon(t-1)] * \frac{\Delta t}{2} \quad (6.4a)$$

where

- $p(t)$ = Current Controller Output
- $p(t-1)$ = Previous Controller Output
- $\varepsilon(t)$ = Current Error Signal
- $\varepsilon(t-1)$ = Previous Error Signal
- Δt = Time Interval between consecutive signals

Equation 6.4a takes the average reading of the current and previous error signal and determines the incremental change in area under the error-time curve by multiplying this average by the time interval between consecutive readings. Instead of using the average value, some systems simply use the current signal, but this is a less accurate approximation of the change in area and it may mislead novices to assume the control equation represents Proportional control when in fact, it is Integral control. Table 6.2 shows example data to demonstrate that these two equations (6.3 and 6.4a) are equivalent. The data are plotted in Fig. 6.3 to show this comparison in graphical form.

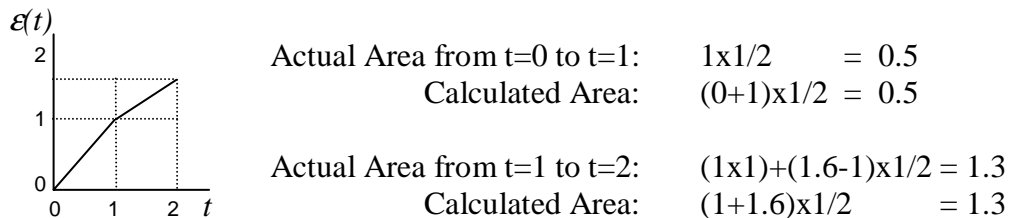


Fig. 6.2. Demonstration that the actual digital area is equivalent to the calculated area.

Table 6.2. Example data to demonstrate the equivalency of the difference form (Eq. 6.4a) to that of Eq. 6.3 for Integral control.

(where $T_I = 2$; $p_o = 325$; $A = 1$; $\Delta t = 50$)

Step change: $\varepsilon(t) = A$ where $A = 1.0$

t	$t-1$	$\varepsilon(t)$	$\varepsilon(t-1)$	$p(t)$ from Eq. 6.4a	$p(t)$ from Eq. 6.3
0	-1	1	0	325	337.5
1	0	1	1	350	362.5
2	1	1	1	375	387.5
3	2	1	1	400	412.5
4	3	1	1	425	437.5
5	4	1	1	450	462.5

Ramp change: $\varepsilon(t) = At$ where $A = 1.0$

t	$t-1$	$\varepsilon(t)$	$\varepsilon(t-1)$	$p(t)$ from Eq. 6.4a	$p(t)$ from Eq. 6.3
0	-1	0	0	325	325
1	0	1	0	350	350
2	1	2	1	425	425
3	2	3	2	550	550
4	3	4	3	725	725
5	4	5	4	950	950

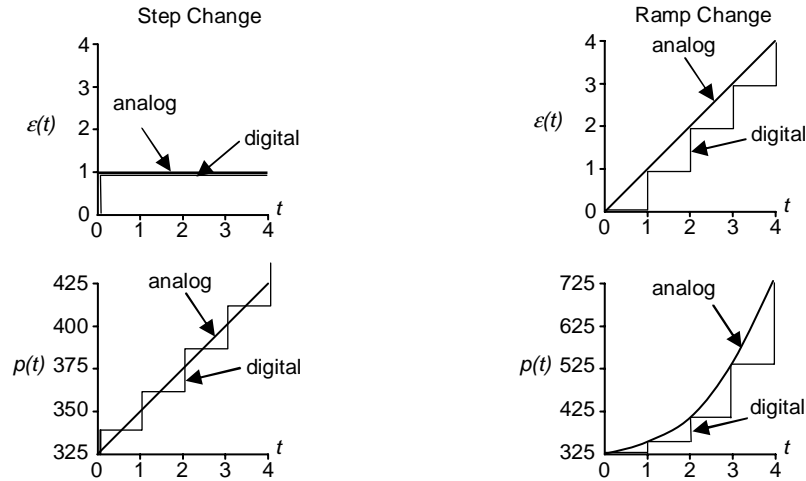


Fig. 6.2. Integral controller response to an analog or digital error signal change.

Derivative Control

Recall that the analog relationship from Eq. 3.2 for Derivative Control is:

$$p(t) = T_D \frac{d\varepsilon(t)}{dt} + p_o \quad (6.5)$$

where $p(t)$ = Controller Output
 $\varepsilon(t)$ = Error signal
 p_o = Initial value of the Controller Output

The initial value with a digital signal cannot be established, so instead one takes the change in slope:

$$p(t) - p(t-1) = T_D \left[\frac{\varepsilon(t) - \varepsilon(t-1)}{\Delta t} - \frac{\varepsilon(t-1) - \varepsilon(t-2)}{\Delta t} \right] \quad (6.6)$$

or by rearranging

$$p(t) = p(t-1) + \frac{T_D}{\Delta t} \varepsilon(t) - \frac{2T_D}{\Delta t} \varepsilon(t-1) + \frac{T_D}{\Delta t} \varepsilon(t-2) \quad (6.6a)$$

where

$p(t)$ = Current Controller Output
 $p(t-1)$ = Previous Controller Output
 $\varepsilon(t)$ = Current Error Signal
 $\varepsilon(t-1)$ = Previous Error Signal
 Δt = Time Interval between consecutive signals

Equation 6.6a takes the difference in the current slope and previous slope of the error-time curve and multiplies it by the Derivative Time Constant. So this results in a time-series equation that uses the current, previous, and past-previous errors. Table 6.3 shows example data to demonstrate that these two equations (6.5 and 6.6a) are equivalent. The data are plotted in Fig. 6.4 to show this comparison in graphical form.

Table 6.3. Example data to demonstrate the equivalency of the difference form (Eq. 6.6a) to that of Eq. 6.5 for Derivative Control.

(where $T_D = 25$; $p_o = 325$; $A = 50$; $\Delta t = 50$)

Step change: $\varepsilon(t) = A$ where $A = 50$

t	$t-1$	$\varepsilon(t)$	$\varepsilon(t-1)$	$\varepsilon(t-2)$	$p(t)$ from Eq. 6.6a	$p(t)$ from Eq. 6.5
0	-1	1	0	0	350	∞
1	0	1	1	0	325	325
2	1	1	1	1	325	325
3	2	1	1	1	325	325
4	3	1	1	1	325	325

Ramp change: $\varepsilon(t) = At$ where $A = 1.0$

t	$t-1$	$\varepsilon(t)$	$\varepsilon(t-1)$	$\varepsilon(t-2)$	$P(t)$ from Eq.6.6a	$p(t)$ from Eq. 6.5
0	-1	0	0	0	350	325
1	0	1	0	0	350	350
2	1	2	1	0	350	350
3	2	3	2	1	350	350
4	3	4	3	2	350	350

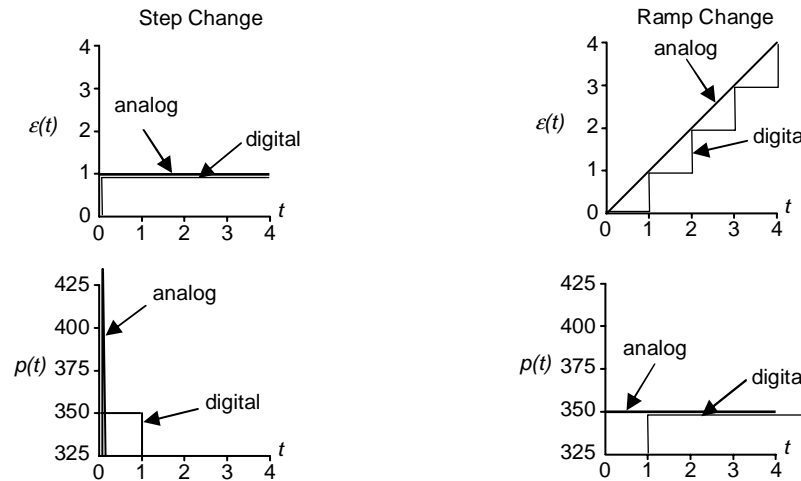


Fig. 6.4. Derivative Controller response to an analog or digital error signal change.

Note that the response of the digital Derivative Controller is slower than that of the analog controller by Δt . As well, the response to a step change does not produce an impulse, but instead a square wave of amplitude equal to T_D is the controller output.

The Digital Equation Form for PID Control

Instead of using K_c , T_I , and T_D parameters with the error signal, it is much more convenient to express the control equation in the form of a time-series with the current, previous, and past-previous errors.

$$p(t) = p(t-1) + K_1 * \epsilon(t) + K_2 * \epsilon(t-1) + K_3 * \epsilon(t-2) \quad (6.7)$$

where

$$K_1 = +K_c + \Delta t / 2T_I + T_D / \Delta t$$

$$K_2 = -K_c + \Delta t / 2T_I - 2T_D / \Delta t$$

$$K_3 = +T_D / \Delta t$$

For quarter decay ratios for a first order process with a pure time delay, using the Cohen and Coon design method, the values of K_1 , K_2 , and K_3 are as follows:

$$K_1 = (A + \Delta t / BT_d + C K_p T_d / \Delta t) / K_p$$

$$K_2 = (-A + \Delta t / BT_d - 2CK_p T_d / \Delta t) / K_p$$

$$K_3 = +CT_d / \Delta t$$

where

$$A = (16 + 3R) / 12R$$

$$B = (64 + 12R) / (13 + 8R)$$

$$C = 4 / (11 + 2R)$$

and $R = T_d / T_p$

with Zeigler-Nichols Rules
$A = 1.2/R$
$B = 4.0$
$C = 0.5$
$R = T_d / T_p$

Table 6.4. Comparison of Cohen-Coon and Zeigler-Nichols Open Loop Rules for different values of $R = T_d/T_p$

Method	Coefficient	$R = T_d/T_p$							
		0	0.1	0.2	0.4	0.6	0.8	1.0	1.2
Cohen-Coon	A	∞ (max)	13.6	6.9	3.6	2.5	1.9	1.6	1.4
	B	4.9	4.7	4.5	4.2	4.0	3.8	3.6	3.5
	C	0.36	0.36	0.35	0.34	0.33	0.32	0.31	0.30
Zeigler-Nichols	A	∞ (max)	12.0	6.0	3.0	2.0	1.5	1.2	1.0
	B	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0
	C	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

What can be seen here is that Cohen and Coon use a higher relative proportional control effect compared to Zeigler-Nichols for all sensible values of R. They recommend using slightly lower relative integral control effect compared to Z-N for values of R below 0.6 and a slightly higher relative control effect for values of R above this limit. Finally, for all sensible values of R Cohen and Coon recommend a lower relative derivative control effect in comparison to Z-N.