MINE 432: Industrial Automation and Robotics in Mining

Lecture 8B

Frequency (or Steady State) Response of First and Second Order Systems

The most interesting type of system response is called the Frequency Response. Unlike the response of a process to a step change, this mode of study uses a sine wave input in which the analysis is interested in the steady-state response rather than the transient response. The steady-state response to a step change is rather trivial and provides information only about the process gain. It provides no knowledge about the delay time or process time constant. The transient response to a sine wave change involves a relatively quick move from the initial value to a consistent response that oscillates at the same frequency as the input.

There are two major characteristics of the frequency response that must be studied: first, the amplitude of the output is either attenuated or amplified depending on the frequency of the forcing function (the sine wave); and secondly the output sine wave oscillates out of phase from the input oscillations with a phase lag or lead also dependent on the forcing function frequency.

First Order Frequency Response

For a sine wave input to a first order process

$$\theta_i(s) = \frac{\omega}{s^2+\omega^2} \tag{8.6}$$

the output is equal to

$$\theta_o(s) = \frac{K_p}{T_p s+1}\cdot\frac{\omega}{s^2+\omega^2} = \frac{A_0}{T_p s+1} + \frac{A_1 s}{s^2+\omega^2} + \frac{A_2}{s^2+\omega^2} \tag{8.7}$$

By partial fraction decomposition, we get:

$$A_0 = \frac{\omega K_p T_p{}^2}{\omega^2 T_p{}^2+1} \quad ; \quad A_1 = \frac{-\omega K_p T_p}{\omega^2 T_p{}^2+1} \quad ; \quad A_2 = \frac{\omega K_p}{\omega^2 T_p{}^2+1} \tag{8.8}$$

and inverting back to the time domain gives:

$$\theta_o(t) = \overbrace{\frac{K_p \omega T_p}{\omega^2 T_p{}^2+1} e^{-t/T_p}}^{\text{dies out for large t}} + \frac{K_p}{\sqrt{\omega^2 T_p{}^2+1}} sin(\omega t+\phi) \tag{8.9}$$

$$\phi = -tan^{-1}(\omega T_p)$$

note: ϕ is not a function of t but rather, of T_p and ω.

So for values of t/T_p equal to 4.0 ($e^{-4} = 0.018$) or greater, the process output, $\theta_o(t)$, is a sine wave attenuated by $\frac{1}{\sqrt{\omega^2 T_p{}^2+1}}$ whose value depends on whether ω is fast (large) or slow (small).

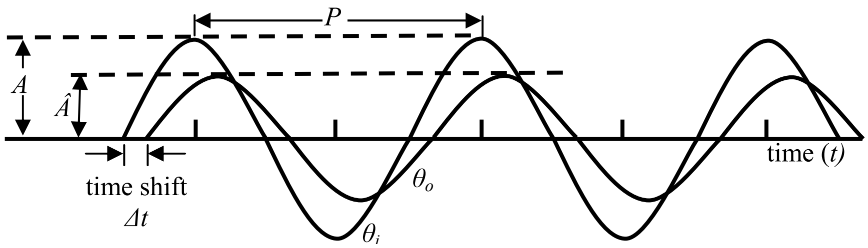


Figure 8B.1. Attenuation and time shift between input and output sine waves (K_p=1). The phase angle ϕ of the output signal is calculated from $\phi = -180*\Delta t/P$, where Δt is the time shift and P is the period of oscillation.

To establish the effect of input sine wave frequency on amplitude attenuation and time shift, the process is tested at a variety of frequencies to build up data over about 3-4 orders of magnitude. The amplitude ratio and phase angle are plotted against input frequency (ωT_p) on a log-log and semi-log plot respectively, and referred to as Bode Diagrams after the person who proposed this analysis method. The Bode diagram for a first order system is shown in Figure 8B.2.

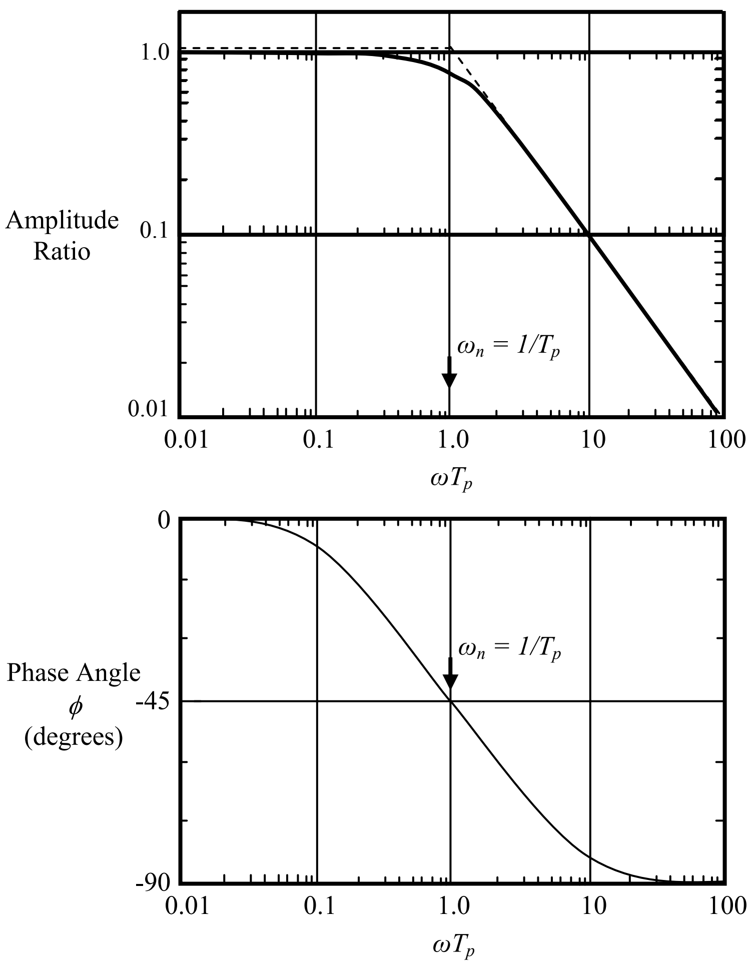


Figure 8B.2. First order system Bode diagram.
(Note that $\omega_n = 1/T_p$, so the X-axis is the relative frequency ω/ω_n).

Why Frequency Response Analysis is so Useful

The real power behind frequency response relates to the nature of the response (an amplitude ratio and a phase lag) and the utility of its application to the Laplace Transform. The technique can be used effectively to study the following aspects of a control system:

- System identification
- Controller tuning
- Stability analysis
- Robustness analysis
- Designing noise filters

Consider a complex number W defined by $a + bj$ where a represents the real part of W or $Re(W)$ and b represents the imaginary part of W or $Im(W)$. We can define the following terms to characterize this complex number in the complex domain (polar coordinates):

1. The *Modulus* (or *Absolute Value* or *Magnitude*) of W:

$$|W| = \sqrt{\left[\mathrm{Re}(W)^2 + \mathrm{Im}(W)^2\right]} \tag{8.10}$$

2. The *Argument* (or *phase angle*) of W:

$$\angle W = tan^{-1}\left[\frac{\mathrm{Im}(W)}{\mathrm{Re}(W)}\right] \tag{8.11}$$

These two terms can be used to plot the complex number in the polar coordinate system as shown in Figure 8B.3.

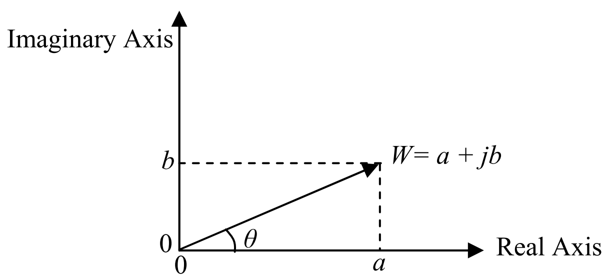


Fig. 8B.3. Polar coordinate system and complex numbers.

From Figure 8B.3, it is clear that

$$a = |W|\cos\theta \qquad \text{and} \qquad b = |W|\sin\theta$$

and

$$W = |W|\cos\theta + j\,|W|\sin\theta$$

Now since $\cos\theta = (e^{+j\theta} + e^{-j\theta})/2$ and $\sin\theta = (e^{+j\theta} - e^{-j\theta})/2j$,

then $W = |W|\,[(e^{+j\theta} + e^{-j\theta})/2] + j\,|W|\,[(e^{+j\theta} - e^{-j\theta})/2j] = |W|\,e^{+j\theta}$ (8.12)

Now let's apply this approach to the first order system previously derived into the time domain:

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{K_p}{T_p s+1} \quad >>>>> \quad \frac{\theta_o(j\omega)}{\theta_i(j\omega)} = \left[\frac{K_p}{j\omega T_p+1}\right]\left[\frac{-j\omega T_p+1}{-j\omega T_p+1}\right] \tag{8.13}$$

By simplifying, we get:

$$\frac{\theta_o(j\omega)}{\theta_i(j\omega)} = \left[\frac{K_p}{\omega^2 T_p^2 + 1}\right] - j\left[\frac{K_p \omega T_p}{\omega^2 T_p^2 + 1}\right] \tag{8.14}$$

Or, in polar coordinate form:

$$\frac{\theta_o(j\omega)}{\theta_i(j\omega)} = \left|\frac{K_p}{\omega^2 T_p^2 + 1}\right| \angle \tan^{-1}(-\omega T_p) \tag{8.15}$$

This result (compared to Eq. 8.9) shows that by substituting $j\omega$ for s in a transfer function and then rearranging the transform into a complex number, we can directly obtain the amplitude ratio and phase angle from the *modulus* and *argument* of the complex number without solving the transform for a sine wave input and inverting it back into the time domain.

Second Order Frequency Response

In studying the frequency response of a second order system, the same technique can be used to determine its response characteristics directly from its Laplace Transform.

So we substitute $s = j\omega$ into the transformed equation as follows:

Original second order transform:

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{K}{T_p^{\,2} s^2 + 2\zeta T_p s + 1} \tag{8.16}$$

Now, substitute $s = j\omega$ and manipulate the terms to get an expression in the form $(a - bj)$:

$$\frac{\theta_o(j\omega)}{\theta_i(j\omega)} = \frac{K}{T_p^{\,2}(j\omega)^2 + 2\zeta T_p j\omega + 1}$$

$$\frac{\theta_o(j\omega)}{\theta_i(j\omega)} = \frac{K}{-\omega^2 T_p^{\,2} + 1 + j2\zeta\omega T_p}$$

$$\frac{\theta_o(j\omega)}{\theta_i(j\omega)} = \frac{K}{-\omega^2 T_p^{\,2} + 1 + j2\zeta T_p \omega}\left[\frac{(-\omega^2 T_p^{\,2} + 1) - j2\zeta\omega T_p}{(-\omega^2 T_p^{\,2} + 1) - j2\zeta\omega T_p}\right] \tag{8.11}$$

$$\frac{\theta_o(j\omega)}{\theta_i(j\omega)} = \frac{K(\omega^2 T_p^{\,2} - 1)}{(1 - \omega^2 T_p^{\,2})^2 + (2\zeta\omega T_p)^2} - j\frac{2K\zeta\omega T_p}{(1 - \omega^2 T_p^{\,2})^2 + (2\zeta\omega T_p)^2}$$

So the steady state response can be determined directly as:

$$A.R. = \left|\frac{\theta_o(j\omega)}{\theta_i(j\omega)}\right| = \frac{K}{\sqrt{(1 - \omega^2 T_p^{\,2})^2 + (2\zeta\omega T_p)^2}} \quad \text{and} \quad \phi = \tan^{-1}\left(-\frac{2\zeta\omega T_p}{1 - \omega^2 T_p^{\,2}}\right) \tag{8.17}$$

The Bode diagram for the frequency (or steady state) response of a second order system is shown below in Figure 8B.4. Note that the shape of the response is dependent on the damping coefficient. As ζ approaches 0, the amplitude ratio goes to infinity at the natural frequency. This is clearly an unstable response demonstrating that the system will be unstable if a sine wave change is input to the system with a frequency equal to (or close to) the natural frequency.

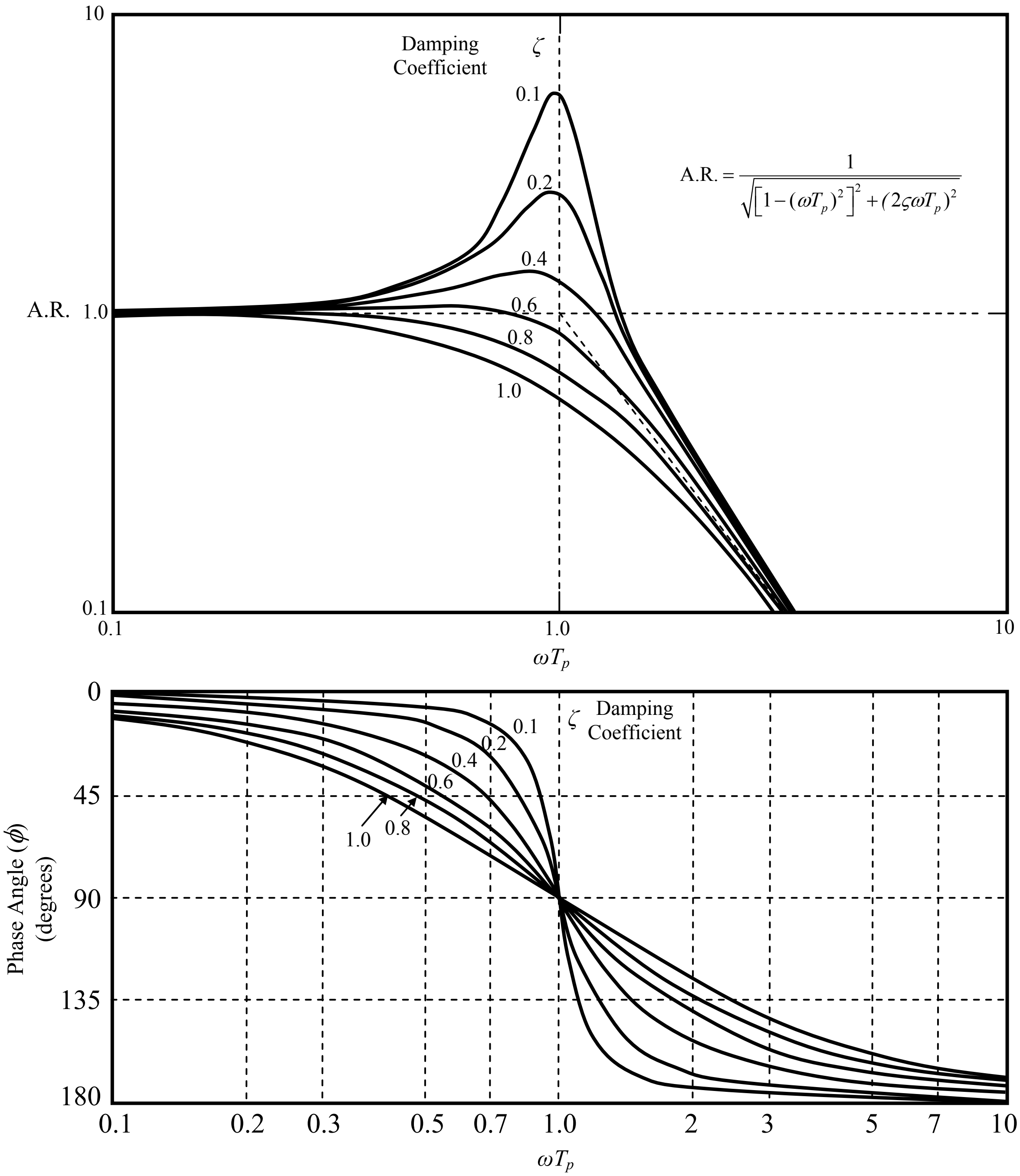


Figure 8B.4. Second order system Bode diagram.
(Note that $\omega_n = 1/T_p$, so the X-axis is the relative frequency ω/ω_n).

Note that a second order system response can lag by as much as 180°, which is twice the maximum lag of a first order system. This is a good way to determine if the process or system is second order, i.e., for high values of ωT_p, if the phase angle lag exceeds 90° then the system is a higher order system. The real problem faced by a second order and higher order systems is the fact that at the natural frequency ω_n, the amplitude ratio can be significantly greater than 1.0 and as the damping coefficient approaches 0, the system response is amplified to infinity. With a 3rd order system, there may be two critical frequencies that must be determined while for a 4th order system there may be three. Clearly such a system response is not tolerable and must be avoided. Knowledge such as this can be used to redesign the system to avoid such excessive behaviour.

As well the phase lag of higher order systems for high frequency inputs is equal to -90n where n is the order of the system. With such high lags, these systems are much more difficult to control and design for than are first order, second order, and dead-time processes.

Frequency Response Characteristics of Controllers

Recall that to find the Frequency Response of any Laplace Transform, we must derive:

1. The amplitude ratio, A.R., and
2. The phase angle, ϕ, which is generally a lag.

For any Transfer Function, *G(s)*:

$$AR = |G(j\omega)|$$
$$\phi = \angle G(j\omega)$$

Proportional Controller

$$G_C(s) = K_C$$

$AR = |K_C|$, and

$$\phi = 0$$

Proportional-Integral (PI) Controller

$$G_C(s) = K_C\left(1 + \frac{1}{T_i s}\right)$$

$$\text{A.R.} = K_C\sqrt{\frac{1}{\omega^2 T_i^2} + 1}$$

$$\phi = tan^{-1}\left(-\frac{1}{\omega T_i}\right)$$

The Bode plot for a PI controller is shown in Figure 8B.5. Note that $\omega_n = T_i^{-1}$ and the asymptotic slope on a log-log plot is – 1 as $\omega \to 0$.

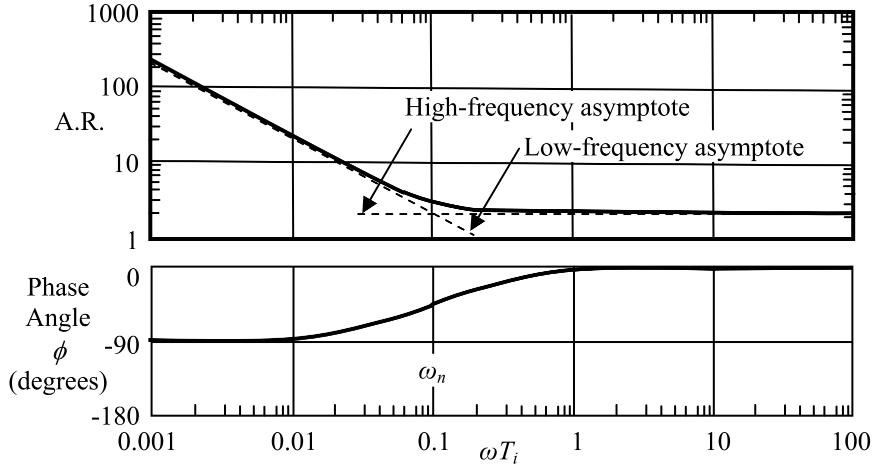


Figure 8B.5. Bode diagram for PI controller where $K_c = 2$ and $T_i = 10$.

Proportional-Derivative (PD) Controller

$$G_C(s) = K_C\left(1 + T_D s\right)$$

$$AR = K_C\sqrt{\omega^2 T_D^{\ 2} + 1}$$

$$\phi = tan^{-1}\left(\omega T_D\right)$$

Note that $\omega_n = T_D^{-1}$ and the high frequency asymptote as $\omega \to \infty$ has a slope of +1 on a log-log plot as in Figure 8B.6.

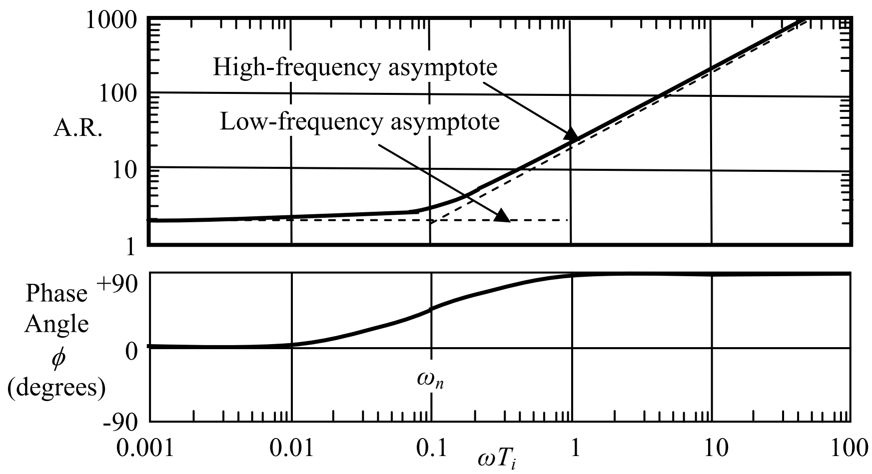


Figure 8B.6. Bode diagram for PD controller where $K_c = 2$ and $T_D = 10$.

Proportional-Integral-Derivative (PID) Controller

$$G_C(s) = K_C\left(1 + \frac{1}{T_I s} + T_D s\right)$$

$$\text{A.R.} = K_C\sqrt{\left(\omega T_D - \frac{1}{\omega T_I}\right)^2 + 1}$$

$$\phi = \tan^{-1}\left(\omega T_D - \frac{1}{\omega T_I}\right)$$

See Figure 8B.7 which includes the asymptotes for both PI and PD controllers.

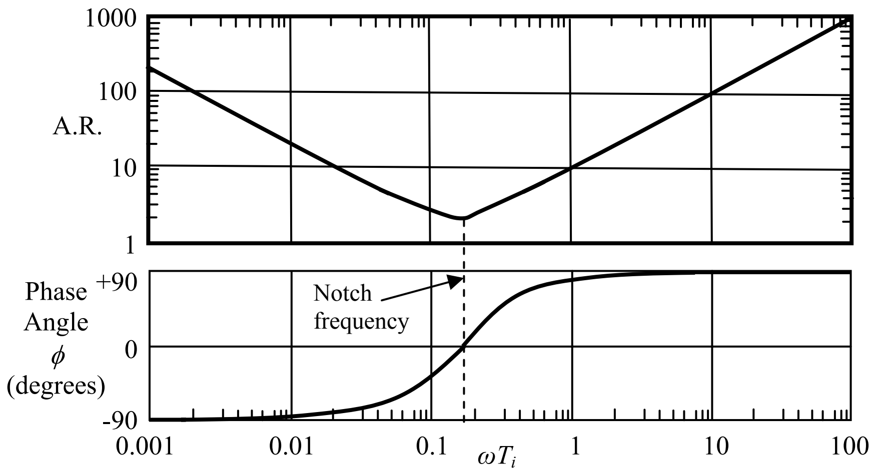


Figure 8B.7. Bode Diagram of a PID controller, $G_C(s) = 2\left(1 + \frac{1}{10s} + 4s\right)$.

Controller Design Using Frequency Response Criteria
The advantages of Frequency Response Analysis are:

1. It is applicable to the dynamic model of any order (including non-polynomials).
2. A Designer can specify the desired closed-loop response characteristics.
3. Information on stability and sensitivity are provided.

The disadvantages of Frequency Response Analysis are:

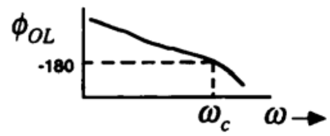
The approach tends to be iterative and hence, is time-consuming–interactive computer graphics are desirable. In addition, results are often taken by mathematically-inclined engineers and technicians to be "gospel" which is far from the reality of most processes and plants. The diagrams represent the expected results for a model–not necessarily for the real process.

Frequency Response Stability Criteria
There are two principal methods used to derive knowledge about a system's stability:

- Bode Stability Criterion
- Nyquist Stability Criterion

The Bode Stability Criterion

The Bode stability criterion states that a closed-loop system is unstable if the Frequency Response of the open-loop Transfer Function $G_{OL}(s) = G_C G_V G_P G_M$ has an amplitude ratio above unity at the critical frequency ω_C. Otherwise the closed-loop system is stable. Note, for stability analysis, $\omega_C \equiv$ value of ω where the open-loop phase angle is -180^0:



- The Bode Stability Criterion provides info on the closed-loop stability from the open-loop frequency response.
- Physical Analogy: Pushing a child on a swing or bouncing a ball.

Example:

A process has a T.F.:

$$G_p(s) = \frac{2}{(0.5s+1)^3} \text{ and } G_V = 0.1, G_M = 10 .$$

For proportional control, determine the closed-loop stability for three values of K_c: 1, 4, and 20.

Solution:

The Open Loop Transfer Function is $G_{OL}(s) = G_C G_V G_P G_M$ or,

$$G_{OL}(s) = \frac{2K_C}{(0.5s+1)^3}$$

The Bode plots for the three values of K_C are shown in Figure 8B.8. Note that the phase angle curves are identical for all three cases. (Explain this?) From the Bode diagrams:

K_c	AR_{OL}	STABLE?
1	0.25	yes
4	1.00	conditionally stable
20	5.00	no

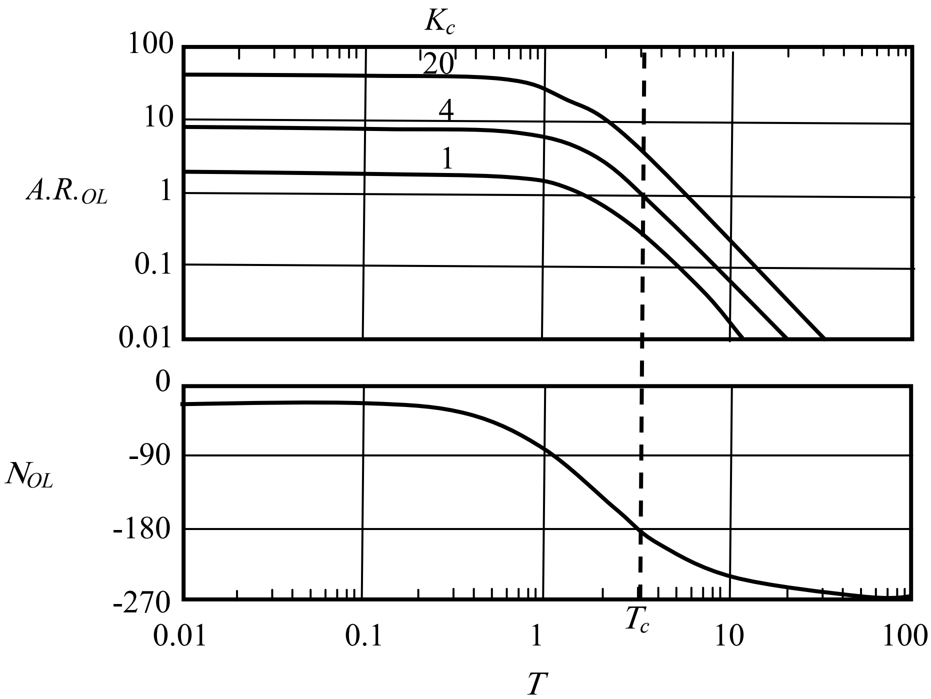


Figure 8B.8. Bode diagrams for $G_{OL}(s) = 2K_c/(0.5s+1)^3$

Example

Determine the closed-loop stability of the system

$$G_p(s) = \frac{4e^{-2}}{5s+1}$$

where $G_V = 2.0, G_M = 0.25$ and $G_c = K_c$. Find ω_C from the Bode diagram. What is the maximum value of K_c for a stable system?

Solution

The Bode plot for $K_c = 1$ is shown in Figure 8B.9. Note that:

$$\omega_C = 1.69 \ \text{rad/min}$$

$$AR_{OL_{\omega=\omega_c}} = 0.235$$

$$\therefore K_{c_{MAX}} AR_{OL} = \frac{1}{0.235} = 4.25$$

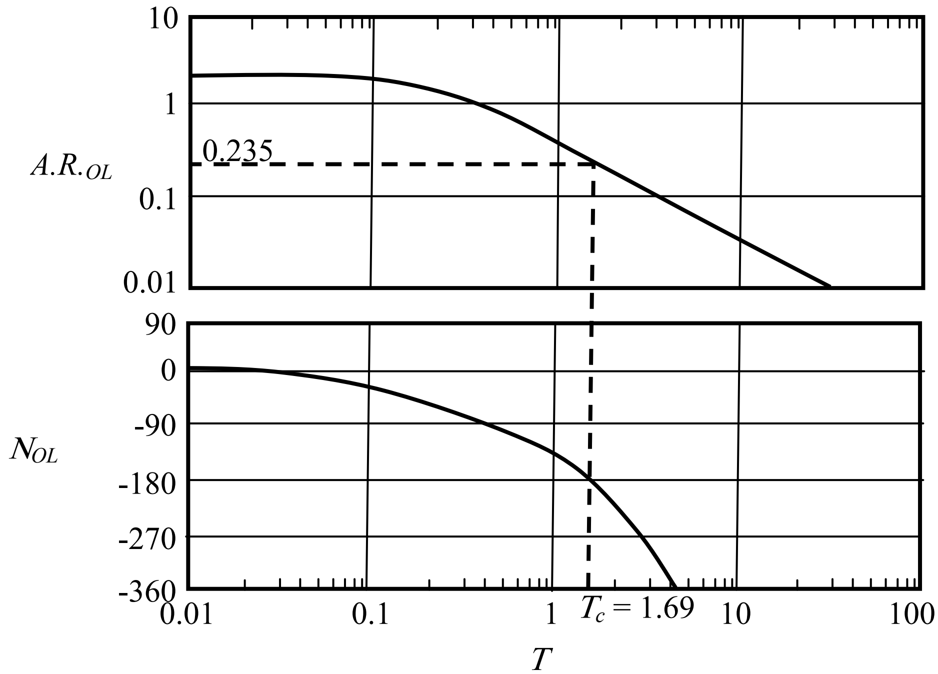


Figure 8B.9. Bode Diagram for $G_{OL}(s) = \dfrac{2K_c e^{-2}}{5s+1}$.

Ultimate Gain and Ultimate Period

The Ultimate Controller Gain, K_{cU}, is the maximum value of $|K_c|$ that results in a stable closed-loop system with a proportional-only controller.

The Ultimate Period is the reciprocal of ω_c as follows: $P_U \equiv \dfrac{2\pi}{\omega_c}$

The value of K_{cU} can be determined from the Open Loop Frequency Response of the system with a proportional-only controller having K_c =1. Thus

$$K_{cU} = \frac{1}{AR_{OL}\big|_{\omega=\omega_c}} \qquad \text{for } K_c = 1$$

Gain and Phase Margins

There are two terms that can be deduced from the Bode Diagram to provide a good design rule to avoid instability. These are known as the gain margin and phase margin. The gain margin (GM) and phase margin (PM) provide measures of how close a system will approach instability. So in the event that the phase angle or amplitude ratio change due to an external variable, these terms can be used to provide a "buffer" to ensure that the system does not go unstable.

Gain Margin
The gain margin is determined from the Amplitude Ratio plot of the Bode Diagram. First determine the critical frequency at which the phase lag equals 180°.

Then let $A_c \equiv AR_{OL}$ at $\omega = \omega_c$.

The gain margin is then defined as: $GM \equiv \dfrac{1}{A_c}$

The Bode Stability Criterion states that if the value of the gain margin is below 1.0 at a phase lag of 180° then the system will be stable for all input sine wave frequencies.

Phase Margin
The phase margin works in the opposite sense. First find the frequency ω_g where the value of AR_{OL} = 1.0, then determine from the phase lag plot of the Bode Diagram what is the corresponding value of the phase lag ϕ_1.

The phase margin is then defined as: $PM \equiv 180^\circ + \phi_1$ since according to the Bode Stability Criterion, if the phase margin is less than 0 then the system will be stable for all sine wave input frequencies. See Figure 8B.10.

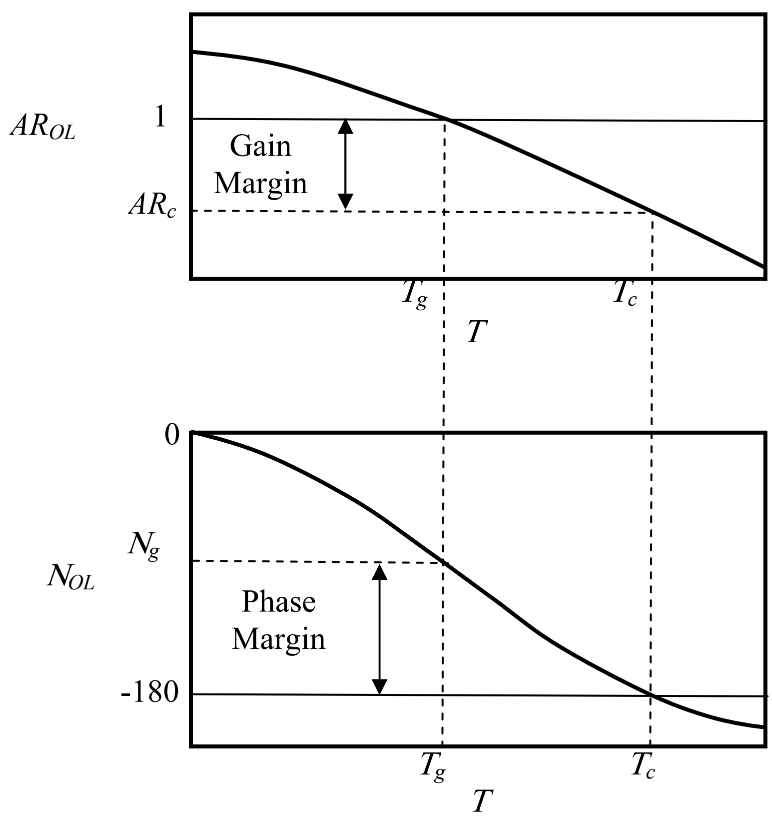


Figure 8B.10. Derivation of the gain margin and phase margin from the Bode Diagram.

Rules of Thumb:

A well-designed feedback control system will have:

$$1.7 \le GM \le 2.0$$
$$30^\circ \le PM \le 45^\circ$$

So the value of K_c can be calculated to place the Gain Margin and Phase Margin between these ranges respectively. If this value is different for each margin, which one should you choose?

Gain-Bandwidth Criterion:
Another approach is to assume that a well-designed control system will maximize the product $K_{cU} \cdot \omega_c$ since for load changes, the value of the Integral of the Absolute Error is calculated as:

$$IAE \cong \frac{C_1}{K_{cu}\omega_c}$$ where C_1 is an integration constant.

So by maximizing this product, the IAE will be minimized.

Closed-Loop Frequency Response Characteristics – transient response
An analysis of the Closed Loop Frequency Response provides useful information about control system performance. As can be seen in Figure 8B.11, although the response achieved by Frequency Response is slower than that using the Zeigler-Nichols method (closed loop), a significantly reduced overshoot is achieved.

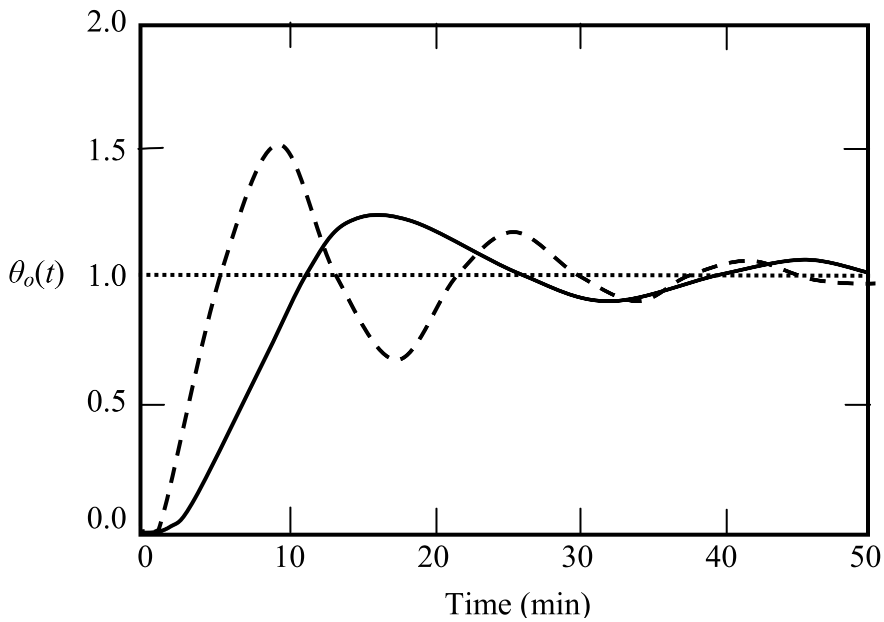


Figure 8B.11. System response to a step change in set point for a PI controller tuned by frequency response (———) and by Zeigler-Nichols rules (– – – –).